Topology and its Applications 31 (1989) 159-168
North-Holland

SECTIONALLY CONTINUOUS INJECTIONS OF EUCLIDEAN SPACES

D.W. CURTIS
Louisiana State University, Baton Rouge, LA 70803, USA

L.R. RUBIN
University of Oklahoma, Norman, OK 73019, USA

Received 7 July 1987

A function $f:\mathbb{R}^n\to\mathbb{R}^n$, $n\geq 2$, is *sectionally continuous* if each restriction $f|H$ to an $(n-1)$-dimensional hyperplane H is continuous. We show that a sectionally continuous injection f is continuous at a point x in $\mathbb{R}^n$ if and only if $f(x)$ is not a limit point of any component of $\mathbb{R}^n\setminus f(\mathbb{R}^n)$. In particular, f is an imbedding if and only if $f(\mathbb{R}^n)$ is open. For $n=2$, we also describe all possible images for sectionally continuous injections with only countably many discontinuities.

AMS (MOS) Subj. Class.: Primary 54C99; secondary 54G20, 57N50

sectionally continuous functions on Euclidean spaces
open imbeddings of Euclidean spaces
Brouwer invariance of domain theorem
Jordan separation theorem

1. Introduction

Professor O. Laback (of the Graz Technical University, Graz, Austria) has asked us the following question. Suppose $f:\mathbb{R}^3\to\mathbb{R}^3$ is a bijection such that for each piecewise-linear arc $K\subset\mathbb{R}^3$, both $f|K$ and $f^{-1}|K$ are piecewise-linear homeomorphisms. Must f be a homeomorphism? In the case of $\mathbb{R}^2$, Laback has shown that every such bijection which preserves the 1-dimensional piecewise-linear structure is a homeomorphism. Motivation for this type of problem comes from his study of space-time structures on manifolds [4, 5], which was influenced by work on weak topologies in [1], [3], and [6].

In this paper we answer Laback's question in the negative; more interestingly, we generalize the positive result for $\mathbb{R}^2$ in several directions. For $n\geq 2$, suppose $f:\mathbb{R}^n\to\mathbb{R}^n$ is an injection such that for each $(n-1)$-simplex $K\subset\mathbb{R}^n, f|K$ is a homeomorphism. Our main result says that f is continuous at a point $x\in\mathbb{R}^n$ if and only if $f(x)$ is not a limit point of any component of $\mathbb{R}^n\setminus f(\mathbb{R}^n)$. In particular, if f is bijective, then it is a homeomorphism. As might be expected, the proof relies heavily on the Jordan Separation Theorem.

2. Properties of sectionally continuous injections

Let $n \geq 2$. We say that a function $f: \mathbb{R}^n \to Y$ is *sectionally continuous* if, for each $(n-1)$-dimensional hyperplane H in $\mathbb{R}^n, f|H$ is continuous. We are primarily interested in sectionally continuous *injections* of $\mathbb{R}^n$ into itself. To permit a more efficient formulation of results, we will usually consider the range space to be the n-sphere $S_n = \mathbb{R}^n \cup \{\infty\}$.

2.1. Proposition. *Let $f: \mathbb{R}^n \to S^n$ be a sectionally continuous injection, and let $S \subset \mathbb{R}^n$ be a piecewise-linear $(n-1)$-sphere, with complementary components U and V. Then $f(S)$ is an $(n-1)$-sphere, with $f(U)$ lying in one component of $S^n \backslash f(S)$ and $f(V)$ lying in the other component.*

Proof. Since S is a finite union of $(n-1)$-simplexes, $f|S$ is continuous. Thus, since f is injective, $f|S: S \to f(S)$ is a homeomorphism. Since the components U and V of $\mathbb{R}^n \backslash S$ are polygonally path-connected, the images $f(U)$ and $f(V)$ are also path-connected, and since they are disjoint from $f(S)$, each of them lies in a component of $S^n \backslash f(S)$. We defer to the end of this section the argument that $f(U)$ and $f(V)$ lie in *opposite* components of $S^n \backslash f(S)$. $\square$

2.2. Proposition. *Let $f: \mathbb{R}^n \to S^n$ be a sectionally continuous injection. Then $S^n \backslash f(\mathbb{R}^n)$ has exactly one compact component, and each noncompact component has a unique limit point in $f(\mathbb{R}^n)$.*

Proof. For each $r > 0$, let $B_r = \{(x_1, \ldots, x_n) \in \mathbb{R}^n: \sup|x_i| \leq r\}$ and $S_r = \text{bd}\, B_r$. Thus, S_r is a piecewise-linear $(n-1)$-sphere. Let U_r denote the component of $S^n \backslash f(S_r)$ containing $f(\text{int}\, B_r)$, and let V_r denote the other component. By the preceding proposition, $f(\text{ext}\, B_r) \subset V_r$. Hence, for each $t > r, f(S_t) \subset V_r$, $\bar{U}_r \subset U_t$, and $\bar{V}_t \subset V_r$.

Consider the continuum $K = \bigcap_{i=1}^{\infty} \bar{V}_i$. Clearly, K is a component of $S^n \backslash f(\mathbb{R}^n)$. Now consider any other component J of $S^n \backslash f(\mathbb{R}^n)$. For some $r > 0$, we must have $J \subset U_r$; let $q = \inf\{r > 0: J \subset U_r\}$. If $q = 0$, $J \subset \bigcap_{i=1}^{\infty} U_{1/i}$. Since $\bigcap_{i=1}^{\infty} \bar{U}_{1/i}$ is a continuum which intersects $f(\mathbb{R}^n)$ in the single point $f(0)$, and since J is a component of $\bigcap_{i=1}^{\infty} \bar{U}_{1/i} \backslash \{f(0)\}$, J must have $f(0)$ as its unique limit point in $f(\mathbb{R}^n)$. If $q > 0$, then $J \subset \bigcap_{i=1}^{\infty} (U_{q+1/i} \cap V_{q-1/i})$. Since $\bigcap_{i=1}^{\infty} (\bar{U}_{q+1/i} \cap \bar{V}_{q-1/i})$ is a continuum whose intersection with $f(\mathbb{R}^n)$ is $f(S_q)$, J must have a limit point in $f(S_q)$, say $f(x)$. Knowing this, we may use the same argument as in the case $q = 0$, replacing the spheres S_r centered at 0 with the corresponding spheres centered at x, to show that $f(x)$ is the unique limit point of J in $f(\mathbb{R}^n)$. This completes the proof of the proposition. $\square$

2.3. Theorem. *Let $f: \mathbb{R}^n \to S^n$ be a sectionally continuous injection, and let $x \in \mathbb{R}^n$. Then f is continuous at x if and only if $f(x)$ is not the limit point of any component of $S^n \backslash f(\mathbb{R}^n)$.*

Proof. The construction given in the proof of the preceding proposition shows that the set $F_0 = \bigcup\{J: J \text{ is a component of } S^n \backslash f(\mathbb{R}^n) \text{ with limit point } f(0)\} \cup \{f(0)\}$ is precisely the continuum $\bigcap_{i=1}^{\infty} \bar{U}_{1/i}$. Since f is continuous at 0 if and only if $\bigcap_{i=1}^{\infty} \bar{U}_{1/i} = \{f(0)\}$, the theorem holds for $x=0$. The same argument applies for arbitrary $x \in \mathbb{R}^n$; simply use spheres in $\mathbb{R}^n$ centered at x. □

The following theorem shows that every sectionally continuous injection is a right inverse of the quotient map of an upper semi-continuous cellular decomposition.

2.4. Theorem. *Let $f: \mathbb{R}^n \to S^n$ be a sectionally continuous injection. Then there exists a continuous surjection $g: S^n \to \mathbb{R}^n \cup \{\infty\}$ with cellular point-inverses such that:*

(i) *for each $x \in \mathbb{R}^n$, $g(f(x)) = x$, and $f(x)$ is path-accessible from $S^n \backslash g^{-1}(x)$;*

(ii) *each component of $S^n \backslash f(\mathbb{R}^n)$ lies in some point-inverse of g; and*

(iii) *f is continuous at $x \in \mathbb{R}^n$ if and only if $g^{-1}(x) = \{f(x)\}$.*

Proof. Let F_∞ denote the compact component of $S^n \backslash f(\mathbb{R}^n)$, and for each $x \in \mathbb{R}^n$, let $F_x = \bigcup\{J : J$ is a component of $S^n \backslash f(\mathbb{R}^n)$ with limit point $f(x)\} \cup \{f(x)\}$. The construction given in the proof of Proposition 2.2 shows that each F_x, $x \in \mathbb{R}^n \cup \{\infty\}$, is a cellular continuum in S^n, i.e., $S^n \backslash F_x \approx \mathbb{R}^n$. (For instance, in the case of $F_\infty = \bigcap_{i=1}^{\infty} \bar{V}_i$ or $F_0 = \bigcap_{i=1}^{\infty} \bar{U}_{1/i}$, it isn't necessary to assume that the $(n-1)$-spheres $f(S_i)$ or $f(S_{1/i})$ are tame in S^n, since in any case they can be approximated by tame spheres; see [2]).

The desired map $g: S^n \to \mathbb{R}^n \cup \{\infty\}$ is defined by setting $g(F_x) = x$, for each $x \in \mathbb{R}^n \cup \{\infty\}$. The continuity of g is clear from the construction of its point-inverses F_x. The image under f of any linear segment in $\mathbb{R}^n$ with one endpoint x is an arc in $(S^n \backslash g^{-1}(x)) \cup \{f(x)\}$ with one endpoint $f(x)$. The other properties claimed for g are obvious. □

2.5. Corollary. *Let $f: \mathbb{R}^n \to S^n$ be a sectionally continuous injection. Then f is a local homeomorphism at $x \in \mathbb{R}^n$ if and only if $f(\mathbb{R}^n)$ is a neighborhood of $f(x)$.*

Proof. Suppose that $f(x) \in U \subset f(\mathbb{R}^n)$, for some open set U. By Theorem 2.3, f is continuous at x. Thus, $f(N(x)) \subset U$ for some neighborhood $N(x)$ of x. Then again by Theorem 2.3, f is continuous over $N(x)$. Since f is injective and $\mathbb{R}^n$ is locally compact, it follows that the restriction of f to some neighborhood of x is an imbedding.

Conversely, suppose that f is continuous, and therefore an imbedding, over some neighborhood $N(x)$ of x. We could invoke the Brouwer Invariance of Domain Theorem to see immediately that $f(N(x))$ must be a neighborhood of $f(x)$. Instead, we give an argument using Theorem 2.4 (thereby obtaining another elementary proof for Invariance of Domain, based on the Jordan Separation Theorem). Suppose that $f(\mathbb{R}^n)$ is *not* a neighborhood of $f(x)$. Then there exists a sequence $\{y_k\}$ in $S^n \backslash f(\mathbb{R}^n)$ converging to $f(x)$. If $g: S^n \to \mathbb{R}^n \cup \{\infty\}$ is the map given by Theorem 2.4, then $\{g(y_k)\}$ converges to $g(f(x)) = x$, and we have some $g(y_k) \in N(x)$. Thus f is continuous at $x_k = g(y_k)$. But this contradicts part (iii) of Theorem 2.4, since $g^{-1}(x_k)$ is non-degenerate. □

2.6. Corollary. *A sectionally continuous injection $f:\mathbb{R}^n \to S^n$ is continuous (and therefore an imbedding) if and only if $f(\mathbb{R}^n)$ is open in S^n. In particular, every sectionally continuous bijection $f:\mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.*

Our final corollary shows that the topological type of any sectionally continuous injection $f:\mathbb{R}^n \to S^n$ is determined by its image.

2.7. Corollary. *Let $f:\mathbb{R}^n \to S^n$ and $f':\mathbb{R}^n \to S^n$ be sectionally continuous injections with the same image. Then there exists a homeomorphism $h:\mathbb{R}^n \to \mathbb{R}^n$ such that $f' = f \circ h$.*

Proof. Let $g: S^n \to \mathbb{R}^n \cup \{\infty\}$ be the map given by Theorem 2.4 (with respect to the sectionally continuous injection f). Note that $g|f(\mathbb{R}^n) = f^{-1}$. Since $f(\mathbb{R}^n) = f'(\mathbb{R}^n)$, and f' is injective, the function $h = g \circ f': \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. And since f' is sectionally continuous, so is h. By the preceding corollary, h is a homeomorphism. Since $g|f'(\mathbb{R}^n) = f^{-1}$, we have $f \circ h = f \circ g \circ f' = f \circ f^{-1} \circ f' = f'$. $\square$

To complete the proof of Proposition 2.1, we need a general, elementary result concerning separation properties in S^n. Let $\Delta \subset S^n$ be an $(n-2)$-sphere, and let D_1, D_2, D_3, and D_4 be closed $(n-1)$-cells in S^n such that Δ is the common combinatorial boundary for each D_i, with $D_i \cap D_j = \Delta$ for all $i \neq j$. We say that the cells D_1 and D_2 lie on the *same side* (respectively, on *opposite sides*) of the $(n-1)$-sphere $D_3 \cup D_4$ if the combinatorial interiors $\mathring{D}_1$ and $\mathring{D}_2$ lie in the same component (respectively, in opposite components) of $S^n \setminus (D_3 \cup D_4)$.

2.8. Lemma. *Let $D_1, D_2, D_3, D_4 \subset S^n$ be $(n-1)$-cells as above, pairwise intersecting in a common combinatorial boundary. Then the following conditions are equivalent*:

(a) *D_2 and D_4 lie on opposite sides of $D_1 \cup D_3$*;

(b) *D_1 and D_4 lie on the same side of $D_2 \cup D_3$, and D_3 and D_4 lie on the same side of $D_1 \cup D_2$.*

Proof. Suppose condition (a) is satisfied. Choose an arc $\alpha \subset S^n$ with endpoints in $\mathring{D}_1$ and $\mathring{D}_4$ such that $\mathring{\alpha}$ lies in the component of $S^n \setminus (D_1 \cup D_3)$ containing $\mathring{D}_4$. Then α is disjoint from D_3, and since D_2 and D_4 lie on opposite sides of $D_1 \cup D_3$, α is also disjoint from D_2. This shows that D_1 and D_4 lie on the same side of $D_2 \cup D_3$. Similarly, choose an arc $\beta \subset S^n$ with endpoints in $\mathring{D}_3$ and $\mathring{D}_4$ such that $\mathring{\beta}$ lies in the component of $S^n \setminus (D_1 \cup D_3)$ containing $\mathring{D}_4$. Then β is disjoint from both D_1 and D_2, hence D_3 and D_4 lie on the same side of $D_1 \cup D_2$.

Conversely, suppose condition (b) is satisfied. Let U_1 denote the component of $S^n \setminus (D_2 \cup D_3)$ which does not contain $\mathring{D}_1$; let U_2 denote the component of $S^n \setminus (D_1 \cup D_3)$ not containing $\mathring{D}_2$; and let U_3 denote the component of $S^n \setminus (D_1 \cup D_2)$ not containing $\mathring{D}_3$. Clearly, U_1, U_2, and U_3 are the components of $S^n \setminus (D_1 \cup D_2 \cup D_3)$. We claim that $\mathring{D}_4 \subset U_2$. Suppose not; then either $\mathring{D}_4 \subset U_1$ or $\mathring{D}_4 \subset U_3$. But $\mathring{D}_4 \subset U_1$ would mean that D_1 and D_4 lie on opposite sides of $D_2 \cup D_3$, while $\mathring{D}_4 \subset U_3$ would mean that D_3 and D_4 lie on opposite sides of $D_1 \cup D_2$. Thus, we must have $\mathring{D}_4 \subset U_2$, which means that D_2 and D_4 lie on opposite sides of $D_1 \cup D_3$. □

We are now in a position to complete the proof of Proposition 2.1. Recall that $f:\mathbb{R}^n \to S^n$ is a sectionally continuous injection, and $S \subset \mathbb{R}^n$ is a piecewise-linear $(n-1)$-sphere. It has already been shown that $f(S)$ is an $(n-1)$-sphere, and that each of the components U and V of $\mathbb{R}^n \backslash S$ is sent by f into a component of $S^n \backslash f(S)$. It remains to be shown that $f(U)$ and $f(V)$ lie in *opposite* components of $S^n \backslash f(S)$. Choose a piecewise-linear $(n-2)$-sphere $\Delta \subset S$, and let D_1 and D_3 denote the $(n-1)$-cells such that $D_1 \cup D_3 = S$ and $D_1 \cap D_3 = \Delta$. Choose piecewise-linear $(n-1)$-cells D_2 and D_4 in $\mathbb{R}^n$ with combinatorial boundary Δ, such that $\mathring{D}_2 \subset U$ and $\mathring{D}_4 \subset V$. Then the cells D_1, D_2, D_3, and D_4 satisfy the intersection hypothesis of Lemma 2.8, with D_2 and D_4 lying on opposite sides of $D_1 \cup D_3$. Thus, D_1 and D_4 lie on the same side of $D_2 \cup D_3$, and D_3 and D_4 lie on the same side of $D_1 \cup D_2$. Now consider the images $D_i' = f(D_i)$, $i = 1, 2, 3, 4$. Since each restriction $f | D_i$ is a homeomorphism, the cells $\{D_i'\}$ also satisfy the intersection hypothesis of Lemma 2.8. The images D_1' and D_4' lie on the same side of $D_2' \cup D_3'$, and D_3' and D_4' lie on the same side of $D_1' \cup D_2'$. Thus, D_2' and D_4' lie on opposite sides of $D_1' \cup D_3' = f(S)$, which means that $f(U)$ and $f(V)$ lie in opposite components of $S^n \backslash f(S)$. □

3. Counterexamples

The examples given below show that the setting of Theorem 2.3 cannot be expanded, either by lowering the dimensional requirement in the definition of sectional continuity, or by dropping the injective part of the hypothesis.

3.1. Example. There exists a bijection $f:\mathbb{R}^3 \to \mathbb{R}^3$ such that for each line L in $\mathbb{R}^3$, both $f|L$ and $f^{-1}|L$ are piecewise-linear homeomorphisms, but f is not continuous.

Proof. Choose a sequence $\{D_i\}$ of disjoint closed disks in $\mathbb{R}^2$ converging to a point p, such that each line in $\mathbb{R}^2$ intersects only finitely many disks D_i. For each i, consider the cylinder $C_i = \{(x, y, z) \in \mathbb{R}^3 \colon (x, y) \in D_i \text{ and } -1 \leqslant z \leqslant 1\}$, and let $h_i\colon C_i \to C_i$ be a piecewise-linear homeomorphism such that $h_i | \text{bd } C_i = \text{id}$ but $d(h_i, \text{id}) \geqslant 1$. Then the desired bijection $f:\mathbb{R}^3 \to \mathbb{R}^3$ may be defined as follows:

$$f(x) = \begin{cases} h_i(x) & \text{if } x \in C_i, \quad i = 1, 2, \dots; \\ x & \text{otherwise.} \end{cases}$$

Since each line L in $\mathbb{R}^3$ intersects only finitely many cylinders C_i, both $f | L$ and $f^{-1} | L$ are piecewise-linear homeomorphisms. However, the facts that $D_i \to \{p\}$ and $d(h_i, \text{id})$ is bounded away from 0 imply that f must be discontinuous at some point $(x, y, z) \in \mathbb{R}^3$ with $(x, y) = p$ and $-1 \leqslant z \leqslant 1$. □

3.2. Example. There exists a sectionally continuous surjection $f:\mathbb{R}^2\to\mathbb{R}^2$ which is not continuous.

Proof. For each $0<m\leqslant 1$, let L_m denote the line through the origin with slope m, and let $f_m: L_m\to L_m$ be the piecewise-linear surjection defined by the following conditions:

(i) $f_m(x, mx)=(x, mx)$ for all $x\leqslant 0$ and for all $x\geqslant 2m$;
(ii) $f_m(m, m^2)=(1, m)$;
(iii) f_m is linear between $(0, 0)$ and (m, m^2); and
(iv) f_m is linear between (m, m^2) and $(2m, 2m^2)$.

Note that $f_1=\mathrm{id}$. The desired function $f:\mathbb{R}^2\to\mathbb{R}^2$ is defined as follows:

$$f(x, y)=\begin{cases} f_m(x, mx) & \text{if } y=mx, \quad 0<m\leqslant 1;\\ (x, y) & \text{otherwise.}\end{cases}$$

Since $f|L_m=f_m$ is continuous for each $0<m\leqslant 1$, and since $f|\mathbb{R}^2\backslash\{(0,0)\}$ is continuous, f is sectionally continuous, but f is discontinuous at $(0, 0)$. □

Remark. Each of the above examples is easily generalized to higher dimensions. Thus for each n, there exists a discontinuous bijection of $\mathbb{R}^n$, all of whose restrictions to $(n-2)$-dimensional hyperplanes are continuous, and there exists a discontinuous surjection of $\mathbb{R}^n$ which is sectionally continuous.

4. Sectionally continuous injections of the plane

In Fig. 1 we describe what is perhaps the simplest example of a sectionally continuous injection $f:\mathbb{R}^2\to\mathbb{R}^2$ with a single discontinuity. The directed paths shown are the images under f of the rays from the point p. We have $f(p)=p$ and $f(\mathbb{R}^2\backslash\{p\})=\mathbb{R}^2\backslash J$, where J is an arc with endpoint p. If the paths are parametrized appropriately, f will be discontinuous only at p.

Theorem 2.4 provides a set of *necessary* conditions for the image of any sectionally continuous injection $f:\mathbb{R}^n\to S^n$, which may be reformulated as follows: there exists

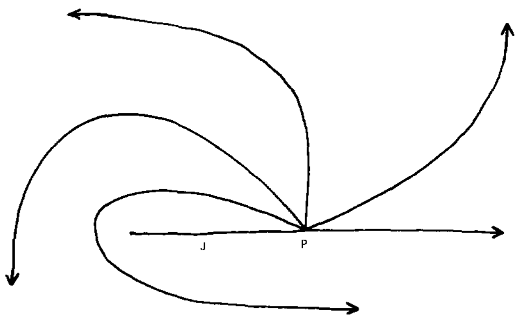

Fig. 1.

an upper semi-continuous cellular decomposition of S^n such that exactly one decomposition element F_∞ is disjoint from $f(\mathbb{R}^n)$; each other decomposition element F_x intersects $f(\mathbb{R}^n)$ in a single point $f(x)$, which is path-accessible from $S^n\backslash F_x$ (in fact, for any sequence $\{x_i\}$ in $\mathbb{R}^n$, $f(x)$ is path-accessible from $S^n\backslash\bigcup_{i=1}^{\infty} F_{x_i}$, as can be seen by considering the image under f of any linear segment in $(\mathbb{R}^n\backslash\{x_1, x_2, \ldots\})\cup\{x\}$ with one endpoint x); each component of $S^n\backslash f(\mathbb{R}^n)$ lies in some decomposition element; and the nondegenerate decomposition elements except F_∞ correspond to the discontinuities of f. In general, we don't know how close, if at all, this set of conditions comes to being *sufficient* for the existence of a sectionally continuous injection $\mathbb{R}^n \to S^n$ with a specified image. What can be shown is that for $n=2$ and for decompositions of S^2 with only countably many nondegenerate elements, the conditions are sufficient. This is a consequence of the following result concerning functions on the plane with only countably many discontinuities.

4.1. Theorem. *Let $f:\mathbb{R}^2\to Y$ be any function such that $Z=\{x\in\mathbb{R}^2: f$ is discontinuous at $x\}$ is countable. Then the following conditions are equivalent*:

(1) *for each $x\in Z$, there exists a decreasing sequence $\{V_n\}$ of connected open sets in $\mathbb{R}^2$ such that $V_n\to\{x\}$ and $f(V_n)\to\{f(x)\}$*;

(2) *for each $x\in Z$, there exists an arc J in $\mathbb{R}^2$ such that $J\cap Z=\{x\}$ and $f|J$ is continuous*;

(3) *there exists a homeomorphism $h:\mathbb{R}^2\to\mathbb{R}^2$ such that $f\circ h:\mathbb{R}^2\to Y$ is sectionally continuous.*

Proof. We leave to the reader the argument that conditions (1) and (2) are equivalent. Suppose f satisfies condition (3), and consider $x\in\mathbb{Z}$. Since $\mathbb{Z}$ is countable, there exists a linear segment K in $\mathbb{R}^2$ such that $K\cap h^{-1}(Z)=\{h^{-1}(x)\}$. Since $f|h(K)$ is continuous, the arc $J=h(K)$ meets the requirements of condition (2). We defer the argument that (2) implies (3); it involves repeated applications of the following lemma.

4.2. Lemma. *Let $p\in\mathbb{R}^2$, let $\alpha\subset\mathbb{R}^2$ be an arc with p as an endpoint, and let V be a neighborhood of $\alpha\backslash\{p\}$. Let $\varepsilon>0$, and let J be a linear segment in $N(p;\varepsilon)$ with p as an endpoint. Let $\sigma:\mathbb{R}^2\backslash J\to(0,1]$ be the map defined by $\sigma(w)=d(w,J)/d(w,p)$. Then there exists a homeomorphism $g:\mathbb{R}^2\to\mathbb{R}^2$ such that*:

(i) *$g(p)=p$ and g is supported on $N(p;\varepsilon)$*;

(ii) *$g(J)\subset\alpha$; and*

(iii) *for each sequence $\{w_n\}$ in $\mathbb{R}^2\backslash J$ converging to p and such that $\{\sigma(w_n)\}$ is bounded away from 0, the sequence $\{g(w_n)\}$ is eventually in V.*

Proof. In constructing g, we use a polar coordinate system (r,θ) for $\mathbb{R}^2$, with p as the pole and the segment J lying on the axis $\theta=0$. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a homeomorphism supported on $N(p;\varepsilon)$ such that $f(p)=p$ and $f(J)\subset\alpha$. Then $f^{-1}(V)$ is a neighborhood of $J\backslash\{p\}=\{(r,0): 0<r\leq\delta\}$, where $\delta=\operatorname{diam} J<\varepsilon$. Let ω: $(0,\delta]\to(0,1)$ be a map such that $\{(r, \theta): 0<r\leq\delta$ and $|\theta|\leq\omega(r)\}\subset f^{-1}(V)$. Let $\rho:(0,\infty)\times[0,2\pi]\to[0,2\pi]$ be a map such that for each $r>0$, $\rho_r=\rho(r,-)$ is a self-homeomorphism of $[0,2\pi]$, and such that for each $r\leq\delta$, ρ_r takes the subinterval $[0,2\pi/(r+1)]$ onto the subinterval $[0,\omega(r)]$, and for each $r\geq\varepsilon$, $\rho_r=\mathrm{id}$. Define a homeomorphism $\tilde\rho:\mathbb{R}^2\to\mathbb{R}^2$ by the formulas $\tilde\rho(r,\theta)=(r,\rho(r,\theta))$ for $r>0$, and $\tilde\rho(p)=p$. Then $\tilde\rho$ is supported on $N(p;\varepsilon)$; $\tilde\rho|J=\mathrm{id}$; and for each sequence $\{w_n\}$ in $\mathbb{R}^2\backslash J$ converging to p and such that $\{\sigma(w_n)\}$ is bounded away from 0, the sequence $\{\tilde\rho(w_n)\}$ is eventually in $f^{-1}(V)$. Thus $g=f\circ\tilde\rho$ is the desired homeomorphism. $\square$

Any homeomorphism g satisfying the conditions of the above lemma will be referred to as a "pinwheel" homeomorphism associated with (p,α,V). Using such homeomorphisms, we now complete the proof of Theorem 4.1. Suppose the function $f:\mathbb{R}^2\to Y$ with a countable discontinuity set $Z=\{z_1,z_2,\dots\}$ satisfies condition (2). Thus for each z_k, there exists an arc α_k in $\mathbb{R}^2$ such that $\alpha_k\cap Z=\{z_k\}$ and $f|\alpha_k$ is continuous. Clearly, there exists a neighborhood V_k of $\alpha_k\backslash\{z_k\}$ such that $f(z)\to f(z_k)$ as $z\to z_k$ through V_k. The desired homeomorphism h will be constructed as an infinite right product $h=\lim_{n\to\infty}g_1\cdots g_n$ of pinwheel homeomorphisms. We describe g_1, g_2, and g_3, from which the continued inductive procedure will be clear.

Let $g_1:\mathbb{R}^2\to\mathbb{R}^2$ be a pinwheel homeomorphism associated with (z_1,α_1,V_1). The choices for the positive constant $\varepsilon=\varepsilon_1$ and the linear segment $J=J_1$ are arbitrary. Then $g_1(z_1)=z_1$, and for each $z\neq z_1$, $g_1(tz+(1-t)z_1)\in V_1$ for all sufficiently small $t>0$. It follows that $f\circ g_1$ is sectionally continuous at z_1. Note that $g_1^{-1}(z_2)\notin J_1$, since $g_1(J_1)\subset\alpha_1$ and $z_2\notin\alpha_1$.

Next, g_2 is chosen to be a pinwheel homeomorphism associated with $(g_1^{-1}(z_2), g_1^{-1}(\alpha_2), g_1^{-1}(V_2))$. We choose $0<\varepsilon_2<d(g_1^{-1}(z_2),J_1)$, and choose any linear segment $J_2\subset N(g_1^{-1}(z_2);\varepsilon_2)$ with endpoint $g_1^{-1}(z_2)$. Then fg_1g_2 is sectionally continuous at $g_1^{-1}(z_2)$ and at $g_1^{-1}(z_1)=z_1$, with $g_2|J_1=\mathrm{id}$. Moreover, we may choose ε_2 sufficiently small so that, for the map $\sigma=\sigma_1:\mathbb{R}^2\backslash J_1\to(0,1]$ used in the lemma, $d(\sigma_1,\sigma_1g_2)<2^{-2}$.

Continuing, we choose a pinwheel homeomorphism g_3 associated with $((g_1g_2)^{-1}(z_3),(g_1g_2)^{-1}(\alpha_3),(g_1g_2)^{-1}(V_3))$. Since $g_1g_2(J_1)\subset\alpha_1$ and $g_1g_2(J_2)\subset\alpha_2$, while $z_3\notin\alpha_1\cup\alpha_2$, we have $(g_1g_2)^{-1}(z_3)\notin J_1\cup J_2$. Choose $0<\varepsilon_3<d((g_1g_2)^{-1}(z_3),J_1\cup J_2)$ sufficiently small so that $d(\sigma_k,\sigma_kg_3)<2^{-3}$ for $k=1,2$. As before, the choice of J_3 is arbitrary. The composition $fg_1g_2g_3$ is sectionally continuous at $(g_1g_2)^{-1}(z_3)$, at $(g_1g_2)^{-1}(z_2)=g_1^{-1}(z_2)$, and at $(g_1g_2)^{-1}(z_1)=z_1$. We have $g_3|J_1\cup J_2=\mathrm{id}$.

Thus, we inductively construct a sequence of homeomorphisms $g_1,g_2,g_3,\dots$. Since each g_n is a uniform homeomorphism, and since the positive constants ε_n may be inductively chosen arbitrarily small, we may obtain a uniform homeomorphism $h=\lim_{n\to\infty}g_1g_2\cdots g_n$.

The fact that $f\circ h$ is sectionally continuous at each point $h^{-1}(z_k)=(g_1\cdots g_{k-1})^{-1}(z_k)$ follows from the properties of the pinwheel homeomorphism g_k and the inductive requirements that $d(\sigma_k,\sigma_kg_n)<2^{-n}$ for each $n>k$. Specifically, consider any ray L with endpoint $h^{-1}(z_k)$, and let $\{x_n\}$ be any sequence on L converging to $h^{-1}(z_k)$. We must show that $f(h(x_n)) \to f(z_k)$. Suppose first that $J_k \subset L$. Then we may assume that $\{x_n\} \subset J_k$, hence $f(h(x_n)) = (fg_1 \cdots g_k)(x_n) \to f(z_k)$ since $g_k(J_k) \subset (g_1 \cdots g_{k-1})^{-1}(\alpha_k)$. Now suppose that $J_k \cap L = \{h^{-1}(z_k)\}$. For each $l > k$, let $h_l = \lim_{i\to\infty}(g_l \cdots g_{l+i})$. Since $\{\sigma_k(x_n)\}$ is bounded away from 0, there exists $l > k$ such that $\{\sigma_k h_l(x_n)\}$ is still bounded away from 0 (here we use the requirement that $d(\sigma_k, \sigma_k g_n) < 2^{-n}$ for each $n > k$). Since $h_l | J_k = \mathrm{id}$, $h_l(h^{-1}(z_k)) = h^{-1}(z_k)$. Thus $h_l(x_n) \to h^{-1}(z_k)$. Since the closure of the support of each of the homeomorphisms $g_{k+1}, \ldots, g_{l-1}$ is disjoint from J_k, we may assume that $(g_{k+1} \cdots g_{l-1})(h_l(x_n)) = h_l(x_n)$ for each n, and therefore $f(h(x_n)) = (fg_1 \cdots g_k \cdots g_{l-1})(h_l(x_n)) = (fg_1 \cdots g_k)(h_l(x_n))$. The fact that $\{\sigma_k h_l(x_n)\}$ is bounded away from 0 implies that $\{g_k(h_l(x_n))\}$ is eventually in $(g_1 \cdots g_{k-1})^{-1}(V_k)$, from which it follows that $(fg_1 \cdots g_k)(h_l(x_n)) \to f(z_k)$. This completes the proof that $f \circ h$ is sectionally continuous.

Theorem 4.1 leads to the construction of sectionally continuous injections of the plane with specified images. In stating the following corollary, we use the fact that a continuum in $\mathbb{R}^2$ is cellular if (and only if) it is nonseparating.

4.3. Corollary. *Let $\mathscr{D}$ be an upper semi-continuous decomposition of $\mathbb{R}^2$ by nonseparating continua such that:*

(i) *$\mathscr{D}$ has only countably many nondegenerate elements $\{D_1, D_2, \ldots\}$; and*

(ii) *each nondegenerate element D_k contains a point p_k which is path-accessible from $\mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} D_n$.*

Then there exists a sectionally continuous injection $f: \mathbb{R}^2 \to \mathbb{R}^2$ with image $f = (\mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} D_n) \cup \{p_1, p_2, \ldots\}$.

Proof. By the classical theorem of R.L. Moore, the quotient space $\mathbb{R}^2/\mathscr{D}$ is homeomorphic to $\mathbb{R}^2$; let $g: \mathbb{R}^2 \to \mathbb{R}^2/\mathscr{D}$ be any homeomorphism. Let $\pi: \mathbb{R}^2 \to \mathbb{R}^2/\mathscr{D}$ be the quotient map, with $\pi(D_k) = w_k$, $k = 1, 2, \ldots$. Let $q: \mathbb{R}^2/\mathscr{D} \to \mathbb{R}^2$ be the injection defined by $\pi q = \mathrm{id}$ and $q(w_k) = p_k$. Let $z_k = g^{-1}(w_k)$. Then the function $\tilde{q} = qg: \mathbb{R}^2 \to \mathbb{R}^2$ is continuous at each $z \notin \{z_1, z_2, \ldots\}$. For each k, let γ_k be an arc in $(\mathbb{R}^2 \setminus \bigcup_1^{\infty} D_n) \cup \{p_k\}$ with p_k as an endpoint. Then $\alpha_k = g^{-1}(\pi(\gamma_k))$ is an arc in $\mathbb{R}^2$ such that $\alpha_k \cap \{z_1, z_2, \ldots\} = \{z_k\}$, and $\tilde{q} | \alpha_k$ is continuous. By Theorem 4.1, there exists a homeomorphism $h: \mathbb{R}^2 \to \mathbb{R}^2$ such that $f = \tilde{q}h = qgh$ is a sectionally continuous injection, with image $f = \text{image } q = (\mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} D_n) \cup \{p_1, p_2, \ldots\}$.

In the above corollary, the full strength of the accessibility condition (ii) is required; it does not suffice to assume only that each point p_k is path-accessible from $\mathbb{R}^2 \setminus D_k$. For each $k = 1, 2, \ldots$, set

$$D_k = (\{-1/k\} \times [-1 - 1/k, 1 + 1/k]) \cup ([-1/k, 1/k] \times \{(-1)^k (1 + 1/k)\})$$
$$\cup (\{1/k\} \times [-1 - 1/k, 1 + 1/k]),$$

and set $J = \{0\} \times [-1, 1]$. Then $\{J, D_1, D_2, \ldots\}$ is the collection of nondegenerate elements of an upper semicontinuous decomposition $\mathscr{D}$ of $\mathbb{R}^2$. Note that while each point of a nondegenerate element is path-accessible from its complement, no point of J is path-accessible from $\mathbb{R}^2\setminus\bigcup_{k=1}^{\infty} D_k$. Thus there can be no sectionally continuous injection $f:\mathbb{R}^2\to\mathbb{R}^2$ whose image intersects each element of $\mathcal{D}$ in exactly one point.

We close with the following questions concerning the possible sizes of the discontinuity sets of sectionally continuous injections.

(1) Does there exist a sectionally continuous injection $f:\mathbb{R}^2\to\mathbb{R}^2$ with an uncountable number of discontinuities? In particular, let C be a Cantor set on the x-axis in $\mathbb{R}^2$. Is there a sectionally continuous injection whose image is the set $\{(x, y)\in\mathbb{R}^2$: either $x\notin C$ or $y\leqslant 0$ or $y>1\}$?

(2) If the answer to (1) is yes, must the set of discontinuities of every sectionally continuous injection $f:\mathbb{R}^2\to\mathbb{R}^2$ have dimension $\leqslant 0$? (An affirmative answer to this question would automatically imply condition (ii) of Theorem 2.4. As it is, condition (ii) already rules out certain situations; for example, if the Cantor set in question (1) is replaced by an interval on the x-axis, then the corresponding subset of $\mathbb{R}^2$ cannot be the image of a sectionally continuous injection).

(3) Do there exist discontinuous, sectionally continuous injections $f:\mathbb{R}^n\to\mathbb{R}^n$ for all $n\geqslant 2$?

References

[1] F.D. Ancel, Collections of paths and rays in the plane which fix its topology, Topology Appl. 15 (1983) 99–110.

[2] F.D. Ancel, Resolving wild imbeddings in dimensions greater than 3, Mathematical Sciences Research Institute, publication 09512-85.

[3] S.P. Franklin and B. Smith-Thomas, Topologies determined by paths, Topology Proc. 4 (1979) 371–384.

[4] O. Laback, On the timelike relation in space-time, Acta Physica Austriaca 52 (1980) 293–299.

[5] O. Laback, Families of arcs which fix the topology of a manifold, preprint.

[6] L.R. Rubin, Line preserving topologies, Gen. Topology Appl. 2 (1972) 193–198.